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Interpolation by Regularized Spline with Tension: II. Application to Terrain Modeling and Surface Geometry Analysis

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Abstract

A general approach to the computation of basic topographic parameters independent of the spatial distribution of given elevation data is developed. The approach is based on an interpolation function with regular first and second order derivatives and on application of basic principles of differential geometry. General equations for computation of profile, plan, and tangential curvatures are derived. A new algorithm for construction of slope curves is developed using a combined grid and vector approach. Resulting slope curves better fulfil the condition of orthogonality to contours than standard grid algorithms. Presented methods are applied to topographic analysis of a watershed in central Illinois.

KEY WORDS : topographic analysis, curvatures, flow lines

INTRODUCTION

Surface geometry analysis plays an important role in the study of various landscape processes. Reliable estimation of topographic parameters, which reflect terrain geometry, is necessary for geomorphological, hydrological, and ecological modeling because terrain controls the fluxes of mass in the landscape. An excellent review of the present state in the development of methods and applications of topographic analysis was recently published by Moore et al. (1991).

Most methods developed up to now have been based on grid digital elevation models with local, often polynomial interpolation used for estimation of first and second order derivatives, which are necessary for computation of topographic parameters (e.g., Evans, 1972; Mark, 1975; Heerdegen and Beran, 1982; Markus, 1986; Franklin, 1987; Papo and Gelbman, 1984; Zevenbergen and Thorne, 1987; Panuska et al., 1991). For construction of slope curves, representing flow lines necessary for computation of flow path length and contributing areas and for simulation of water flow, grid based algorithms usually consider only eight possible directions of flow from each cell that can lead to unrealistic situations and often is not sufficient for preserving the orthogonality of slope curves to contours. The grid-based contour interpolation method (Hutchinson 1988) overcomes some of the usual limitations of grid-based methods by tracing the slope curves on ridges and in valleys exactly from the surface corresponding to the grid.

Another approach uses a triangular irregular network (e.g., Krcho, 1973; Flacke et al., 1990). Triangulation is performed most often in a horizontal plane without considering the position of data points in three dimensional space. Manual interactive manipulation of triangles often must be performed to ensure their correct position according to three dimensional terrain, or additional information on valleys and ridge lines is needed (Mitášová, 1985; Auerbach and Schaeben, 1990).

One of the most comprehensive approaches based on elements bounded by contours and flow trajectories (contour-based approach) was developed by Moore (1988) and was compared with a grid based approach by Panuska et al. (1991). The contour-based approach reflects the spatial distribution of water flow over the terrain and is specially designed to fulfil requirements of hydrologic modeling. However, it requires substantially larger memory and is more difficult to combine directly with other spatial data.

A method that is independent of the spatial distribution of input data (grid, scattered points or digitized contours) and that uses the standard grid structure for storing, manipulating, and visualizing the results of topographic analysis but that at the same time achieves the quality of the contour-based method is presented. It is based on interpolation with a completely regularized spline with tension (part I), which was specially constructed to fulfil the requirements of topographic analysis when reliable estimation of first and second order derivatives is needed.

In the next section the computation of surface geometry parameters from derivatives of the interpolation function is described and general equations for three types of geomorphologically significant curvatures are derived. An application of the approach to real terrain is illustrated with an example from central Illinois.

SURFACE GEOMETRY ANALYSIS

Surface geometry can be analyzed efficiently when the surface is interpolated with a bivariate function $z = f(x,y)$, that is continuous up to second order derivatives and when parameters characterizing surface geometry (topographic parameters) are expressed *via* derivatives of this function. This approach is demonstrated on the computation of basic topographic parameters: slope, aspect, profile curvature, plan curvature, tangential curvature, and on the computation of flow path length.

Before deriving mathematical expressions for these parameters, using the basic principles of differential geometry, the following simplifying notations are introduced:

$$f_x = \frac{\partial z}{\partial x}, \qquad f_y = \frac{\partial z}{\partial y}, \qquad f_{xx} = \frac{\partial^2 z}{\partial x^2}, \qquad f_{yy} = \frac{\partial^2 z}{\partial y^2}, \qquad f_{xy} = \frac{\partial^2 z}{\partial x \partial y} \quad (1)$$

and

$$p = f_x^2 + f_y^2 \ , \quad q = p + 1 \quad . \qquad (2)$$

The steepest slope angle γ and aspect angle α are computed from gradient $\nabla f = (f_x, f_y)$ (its direction is upslope) as follows

$$\gamma = \arctan\sqrt{p} \qquad (3)$$

$$\alpha = \arctan\frac{f_y}{f_x} \qquad (\alpha = 0 \ \text{in west direction}) \quad . \qquad (4)$$

Experience demonstrates that a minimum slope value $\gamma_{min} > 0$ exists under which computation of aspect using equation (4) is inappropriate and the terrain should be classified as flat or as a singular point (peak, pit, or saddle point) with undefined aspect. The minimum slope value depends on the accuracy of elevation data, and the aspect computed in areas with slope less than this value reflects, more likely, 'noise' in the digital elevation model rather than real structure of terrain.

The length of the flow path for each grid point is computed as the length of slope curve generated upslope from this point. Two different types of points on surface are distinguished: regular points, where the magnitude of gradient $|\nabla f| \neq 0$ with one and only one direction of flow, and singular points, where $|\nabla f| = 0$ with undefined direction of flow. The slope curves are constructed only from regular points. To construct the smooth and accurate slope curves efficiently, points defining the slope curve are computed as points of intersection of a line drawn in the gradient direction and a grid cell edge (Fig. 1). Linear interpolation is used to estimate the gradient direction (aspect) at a point on the grid cell edge from values of aspect in neighboring grid points computed from the bivariate interpolation function. The slope curve stops at the cell edge, where slope is less than the given γ_{min}, and the grid cell represents a flat terrain or a singular point. If a singular point is localized inside the cell with slope values greater than the given γ_{min} on its edges, it is found during the process of slope curve generation. In this case, the slope curve begins to cycle due to the configuration of gradient vectors in vertices of the given grid cell, and the slope curve stops at the point where it entered the grid cell with the singular point. This algorithm allows the generation of slope curves that more accurately fulfil the condition of orthogonality to contour lines than do the standard grid based methods.

Computation of curvatures is more complicated because, in general, the surface has different curvatures in different directions and which one is important must be determined according to the type of processes under study. For applications in geosciences, the curvature in gradient direction (profile curvature) is important because it reflects the change in slope angle and thus controls the change of velocity of mass flowing down along the slope curve. The curvature in a direction perpendicular to the gradient reflects the change in aspect angle and influences the divergence/convergence of water flow. This curvature usually is measured in the horizontal plane as the curvature of contours and is called plan curvature (Zevenbergen and Thorne, 1987, Moore et al., 1991). However, for the study of flow divergence/convergence, it is more appropriate to introduce a curvature measured in the normal plane in the direction perpendicular to gradient (Krcho 1973; Krcho 1991). This curvature will be called here a tangential curvature because the direction perpendicular to gradient is, in fact, the direction of tangent to contour at a given point. Equations for these curvatures can be derived using the general equation for curvature κ of a plane section through a point on a surface (Rektorys, 1969)

$$\kappa = \frac{f_{xx}\cos^2\beta_1 + 2f_{xy}\cos\beta_1\cos\beta_2 + f_{yy}\cos^2\beta_2}{\sqrt{q}\ \cos\vartheta} \tag{5}$$

where ϑ is the angle between the normal to the surface at a given point and the section plane; β_1, β_2 are angles between the tangent of the given normal section at a given point and axes x, y , respectively. The equation for profile curvature κ_s at a given point, is computed as the curvature of normal plane section in a gradient direction that means

$$\cos\vartheta = 1, \qquad \cos\beta_1 = \frac{f_x}{\sqrt{p\,q}}, \qquad \cos\beta_2 = \frac{f_y}{\sqrt{p\,q}} \tag{6}$$

and after substitution of (Eqn. 6) to (Eqn. 5) the profile curvature is

$$\kappa_s = \frac{f_{xx} f_x^2 + 2 f_{xy} f_x f_y + f_{yy} f_y^2}{p\sqrt{q^3}} \quad . \tag{7}$$

Similarly, the equation for plan curvature κ_h at a given point is derived as the curvature of horizontal plane section after setting

$$\cos\vartheta = \sqrt{\frac{p}{q}}, \qquad \cos\beta_1 = \frac{f_y}{\sqrt{p}}, \qquad \cos\beta_2 = -\frac{f_x}{\sqrt{p}} \tag{8}$$

and

$$\kappa_h = \frac{f_{xx} f_y^2 - 2 f_{xy} f_x f_y + f_{yy} f_x^2}{\sqrt{p^3}} \quad . \tag{9}$$

The equation for tangential curvature κ_t at a given point is derived as the curvature of normal plane section in a direction perpendicular to gradient (direction of tangent to the contour line) after setting

$$\cos\vartheta = 1, \qquad \cos\beta_1 = \frac{f_y}{\sqrt{p}}, \qquad \cos\beta_2 = -\frac{f_x}{\sqrt{p}} \tag{10}$$

and

$$\kappa_t = \frac{f_{xx} f_y^2 - 2 f_{xy} f_x f_y + f_{yy} f_x^2}{p\sqrt{q}} \quad . \tag{11}$$

From equations (9), (11) easily can be shown that curvatures κ_h and κ_t have identical zero isoline and the spatial distribution of convex and concave areas is the same. However, their values are different because $\kappa_h = \kappa_t / \sin\gamma$ (Krcho, 1973). Similarly as for aspect, equations (7,9,11) are appropriate for computation of curvatures at points with slopes greater than a certain γ_{min} dependent on the accuracy of elevation data; otherwise, the point should be classified as singular or flat.

The positive and negative values of profile and tangential curvature can be combined to define the basic geometric relief forms (Krcho, 1973; Krcho, 1991;

Dikau, 1989). Each form has a different type of flow. Convex and concave forms in gradient direction have accelerated and slowed flow, respectively, and convex and concave forms in tangential direction have converging and diverging flow, respectively (Fig. 2).

In the way presented above, other types of curvatures, such as the principle, mean, or Gauss curvatures as well as curvatures in an arbitrary direction can be computed directly from the interpolation function.

Computation of curvature requires a reliable estimation of second order derivatives which are sensitive to the accuracy of interpolation. An interpolation function, completely regularized spline with tension, has been constructed (Mitášová and Mitáš, part I), that is suitable for topographic analysis because of its accuracy and regular derivatives of all orders.

Topographic analysis, using the presented approach, is performed as follows. For given data points (scattered, digitized contours, or grid), the interpolation function, given by equations (5, 6, 9, 10) in part I, is constructed, and z values as well as values of first and second order derivatives are computed for an output grid with selected resolution. If the data set is large, segmented processing is applied (see part I). Using equations (3, 4, 7, 9, or 11), topographic parameters (slope, aspect, and curvatures) are computed for each grid point. Coordinates of slope curves and length of the flow path are computed, using the algorithm described above. The procedure results in six grid files with values of elevation, slope, aspect, profile, plan/tangential curvatures, and flow path length and one vector file with coordinates of slope curves.

EXAMPLE OF APPLICATION

The approach was used in a study of the environmental impact of a proposed water reservoir in a watershed in central Illinois (Fig. 3). Topographic analysis was necessary for location of areas with the greatest erosion and deposition potential so that proper measures can be taken to minimize siltation in the future reservoir.

Elevation data were provided in the form of digitized contours from USGS quadrangle maps. The spatial distribution of data was strongly heterogeneous, with few data in large, nearly flat areas and dense data along streams. The digital terrain model for the whole watershed under study was computed from approximately 150,000 data points using segmented interpolation with completely regularized spline as described in part I. Because of abundant data along streams, the size of segment was relatively small (120m) and the variable overlapping neighborhood was crucial for successful interpolation. In areas with sparse data the segmentation algorithm had to find the data points for interpolation in several levels of neighboring segments to ensure the smooth connection of interpolated surfaces (part I). Various minimum proximities (10m, 20m, 30m) of input data and grid cell resolutions (20m, 40m, 80m) for the resulting digital elevation model have been compared. Minimum proximity between data points 10m and grid cell size 20m were chosen as appropriate for preserving details of the topographic map and to support the direct combination with satellite imagery (SPOT) data. Values of elevation, slope, aspect, tangential curvature, profile curvature, and length of the flow path were then computed in 1000 x 1600 grid cells for the whole watershed under study.

To illustrate some results in detail a small typical area (500m x 500m) was selected with results computed for grid cells 10m x 10m. A three dimensional view of the terrain in this area with shaded areas representing concavity in the steepest slope direction (Fig. 4) indicates water flow with decreasing velocity. Shaded horizontally concave areas (Fig. 5) in the same view reveals locations with convergent water flow. Slope curves generated upslope from the grid points (Fig. 6) were used for computation of flow path length.

Some results of topographic analysis for a larger part of the watershed (approximately 5000m x 5000m) computed with a resolution of 20m x 20m are illustrated for elevations (Fig. 7), profile curvature (Fig. 8), and tangential curvature (Fig. 9). For presentation at the medium scale, maps were generalized by smoothing (using the function SCAN-majority in ERDAS software described in ERDAS Field Guide, 1990), and the grid was reduced to 40m x 40m resolution for creating the three dimensional view of terrain. The maps clearly show the different pattern of profile and plan curvatures which is the result of the different character of processes underlying their formation.

The three dimensional views of results of topographic analysis presented here were created using the ARC/INFO software (ARC/INFO Surface Modeling and Display,1989). The maps of topographic parameters provide valuable information about the character of terrain in the watershed of the proposed reservoir; computed values have been used in erosion/deposition potential models.

CONCLUSIONS

The approach to topographic analysis based on interpolation by completely regularized spline, specially constructed to meet the requirements of detailed topographic analysis for large areas has been presented and applied to a watershed in central Illinois. Topographic parameters are computed directly from the interpolation function using general equations derived from differential geometry. An improved algorithm for computation of slope curves also is presented. This algorithm produces smooth slope curves that fulfil the condition of orthogonality to contour lines better than standard grid methods.

Presented methods for topographic analysis were incorporated into GRASS (Geographic Resources Analysis Support System) as commands s.surf.tps for interpolation and computation of slope, aspect and curvatures and r.flow (as a contributed code) for slope curve construction and computation of flow path length and slope curves density (GRASS4.1 Reference Manual, 1993).

The approach is by no means exhausted by computation of presented parameters, as derivatives and flow lines can be used for further analysis (singular points, extremes of curvatures, contributing areas, *etc.*) and directly input into various models of landscape processes influenced by topography. The geometry of various types of surfaces representing the spatial distribution of various phenomena can be analyzed within this approach as its application is not restricted to the topography of terrain.

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Figure captions

Fig. 1. Construction of slope curves (thick lines) upslope from grid points A_0 and B_0 using gradients (arrows) computed from the interpolation function. Direction of gradient in point A_i on grid cell edge is linearly interpolated from directions of gradients in grid points C_0 and D_0.

Fig. 2. Basic geometrical relief forms and their type of flow: a) concave profile and tangential curvatures (positive values as computed from Eqn.(7),(11)) with slowed convergent flow; b) convex profile and concave tangential curvatures with accelerated convergent flow; c) concave profile and convex tangential curvatures with slowed and divergent flow; d) convex profile and tangential curvatures (negative values Eqn. (7), (11)) with accelerated divergent flow.

Fig. 3. Map of watersheds in Illinois (study area shown in black), and the view of terrain in the watershed under study. Subregions A and B are shown in detail in Fig. 4-6 and Fig. 7-9 respectively.

Fig. 4. View of terrain in a subregion A (500m x 500m) with shaded areas of profile concavity (slowed flow).

Fig. 5. View of terrain in a subregion A (500m x 500m) with shaded areas of tangential concavity (convergent flow).

Fig. 6. View of terrain in a subregion A (500m x 500m) with slope curves generated upslope from grid points that have been used for computation of flow path length for each grid point.

Fig. 7. View of terrain in 5km x 5km subregion B.

Fig. 8. View of terrain in 5km x 5km subregion B with profile convexity (light shade) and concavity (dark shade).

Fig. 9. View of terrain in 5km x 5km subregion B with tangential convexity (light shade) and concavity (dark shade).