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GENERAL VARIATIONAL APPROACH TO THE INTERPOLATION PROBLEM

L. MITÁŠ
Institute of Physics, EPRC SAS, Dúbravská cesta 9, 842 28 Bratislava, Czechoslovakia

H. MITÁŠOVÁ
Department of Physical Geography and Cartography, Faculty of Natural Sciences, Comenius University, Mlynská dolina, 842 15 Bratislava, Czechoslovakia

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Abstract—The Talmi and Gilat variational approach to the interpolation problem in arbitrary dimension is presented together with the corresponding physical model. The connection of this approach to some known spline methods is demonstrated and new interpolation functions are derived for one-, two- and three-dimensional cases. They are designed to be flexible through the use of meaningful parameters and to give good approximations of both the function itself and its derivatives as well.

1. INTRODUCTION

The interpolation problem is currently met in many disciplines when the studied phenomena is modeled by a continuous function measured only in a few discrete points. The problem is stated as follows: given the N values of studied phenomena $z^{(j)}, j = 1, \ldots, N$ measured at discrete points $x^{(j)} = (x_1^{(j)}, x_2^{(j)}, \ldots, x_d^{(j)}), j = 1, \ldots, N$ within a certain region of a d-dimensional space. Construct a function $S(x)$ so that

$$S(x^{(j)}) = z^{(j)}, \quad j = 1, \ldots, N. \tag{1}$$

It is clear that there exist an infinite number of such functions and to find a unique solution (to ensure the problem to be well posed) additional conditions are introduced. According to various types of these conditions a large number of methods have been proposed, which give more or less satisfactory results in a variety of cases (for the review see, for example Lancaster and Sakauskas [1] de Boor [2], Franke [3]).

A large class of interpolation methods including the splines are based on an intuitively appealing fact that the interpolation function should be smooth. The smoothness condition can be formulated within variational principles as a minimization of the considered smoothness functional. This condition was used in various modifications by several authors, for example Briggs [4], Terzopoulos [5], especially in connection with the spline interpolants presented, for example by Ahlberg *et al.* [6], Duchon [7], Meinguet [8], Dyn and Levin [9], Renka [10] Bini [11]. Probably the most general form of this approach was introduced by Talmi and Gilat [12].

The purpose of this paper is to demonstrate the power of the variational approach in several aspects and to present some new results obtained within this principle. Special attention is paid to the control and interpretation of the interpolants' properties. The connection of this approach to some well known methods is given and new interpolation functions are derived. The problem is solved generally in d-dimensions, practical expressions are given for $d = 1, 2, 3$. The new methods have been designed to meet the requirements of flexibility, sufficient accuracy both for the function itself and its derivatives and reasonable computer implementation.

In Section 2 the physical model for the smooth interpolation is presented and in Section 3 the method is introduced in its general-theoretical form. In Sections 4–7 special cases are studied and some new interpolation functions are proposed. In Section 8 we discuss the applications and extensions of the approach.

2. THE PHYSICAL MODEL FOR THE SMOOTH INTERPOLATION

The fundamental problem of the interpolation is the choice of a suitable additional condition to the interpolation constraints (1) which ensures the problem to be well posed. A clear and useful treatment of this problem can be presented in terms of the following physical model (considered in two dimensions as the most instructive for our purposes).

Suppose we have a thin flexible plate of elastic material that is planar in the absence of external forces and constrain it to pass through the point supports which represent the data points. It is not difficult to imagine that in its equilibrium state the thin plate will generally trace a smooth surface. It is a known fact that the plate minimizes its blending energy what can be—at a certain degree of approximation—expressed through the variational (minimization) condition

$$E_P(S) = \int_{\mathbb{R}^2} \left[\left(\frac{\partial^2 S}{\partial x_1^2}\right)^2 + 2\left(\frac{\partial^2 S}{\partial x_1\,\partial x_2}\right)^2 + \left(\frac{\partial^2 S}{\partial x_2^2}\right)^2\right] \mathrm{d}x_1\,\mathrm{d}x_2 = \text{minimum}, \tag{2}$$

where $\mathbb{R}$ denotes the set of all real numbers. The interpolation function which fulfils this condition is known as a thin plate spline developed by Duchon [7]. This model results in pleasantly smooth surfaces, however problems arise when we have the regions with a rapid change of gradients in the modeled phenomena. Due to the plate's stiffness unacceptable features appear in this case (for example false minimum or maximum).

The stiffness of the plate can be suppressed by assuming a more general model with the variational condition

$$E_{MP}(S) = E_P(S) + \varphi^2 \int_{\mathbb{R}^2} \left[\left(\frac{\partial S}{\partial x_1}\right)^2 + \left(\frac{\partial S}{\partial x_2}\right)^2\right] \mathrm{d}x_1\,\mathrm{d}x_2 = \text{minimum}, \tag{3}$$

where $\varphi \geqslant 0$ is a weight constant. For $\varphi \to 0$ we obtain the model of the thin plate. On the other hand if $\varphi \to \infty$, the resulting function represents the shape of the membrane (rubber sheet) passing through the data points. The membrane does not exhibit the false effects mentioned above but it is not sufficiently smooth for most applications. The function corresponding to condition (3) can be also interpreted as a thin plate with a tension applied to its boundary. A similar idea for one or two dimensions has been used for splines with tension for example by Pruess [13], de Boor [2], Renka [10], Cox [14].

This model performs well until we want to estimate the second derivatives. For this purpose the interpolation surface is not smooth enough, which becomes evident through the fact that the corresponding function exhibits singularities in its second derivatives in the data points. We must improve the stiffness of the plate and this can be done by adding the third (or even higher) derivatives with appropriate weight $\tau \geqslant 0$ to the variational condition

$$E(S) = E_{MP}(S) + \tau^2 E_H(S) = \text{minimum} \tag{4}$$

and

$$E_H(S) = \int_{\mathbb{R}^2} \left[\left(\frac{\partial^3 S}{\partial x_1^3}\right)^2 + \left(\frac{\partial^3 S}{\partial x_2^3}\right)^2 + 3\left(\frac{\partial^3 S}{\partial x_1\,\partial x_2^2}\right)^2 + 3\left(\frac{\partial^3 S}{\partial x_1^2\,\partial x_2}\right)^2\right] \mathrm{d}x_1\,\mathrm{d}x_2. \tag{5}$$

It is interesting to note that for $\tau \ll 1$ the stiffness of the interpolant is changed only in a small neighbourhood of the data point and thus the plate ($\varphi = 0$) or membrane-plate ($\varphi \neq 0$) character of the interpolant is saved. Of course there can exist several applications where the choice $\tau \gg 1$ can be a more preferable one.

In this way we obtain rather general model for the interpolation problem represented by the constrained equilibrium state of a thin flexible plate the stiffness (or smoothness) of which is controlled by the proper choice of the weights in condition (4). It is evident that analoguous models can be proposed also for the one- and three-dimensional cases. The brief mathematical background for this model is given in the following section.

3. FORMULATION OF THE INTERPOLATION PROBLEM AND ITS FORMAL SOLUTION

To introduce the mathematical background in this section we will briefly describe some of the important results of Talmi and Gilat [12]. We start with the definition of the "smoothness co-functional" (inner product) which is the central point of the theory. Let $g(x)$ and $h(x)$ be the elements of the space W of the complex analytic functions of d-real variables defined on some region Ω of the d-dimensional real space ($x = (x_1, x_2, \ldots, x_d)$). Let us further denote by $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d)$ the multiindex with nonnegative integer components, and its length as

$$|\alpha| = \sum_{m=1}^{d} \alpha_m. \tag{6}$$

Then we can express this inner product in the form

$$I(g, h) = \sum_{\alpha} B_\alpha \int_\Omega \cdots \int \left[\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_d^{\alpha_d}} g(x)\right]^* \left[\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_d^{\alpha_d}} h(x)\right] \mathrm{d}x, \tag{7}$$

where

$\mathrm{d}x = \mathrm{d}x_1\, \mathrm{d}x_2 \ldots \mathrm{d}x_d$
B_α = some nonnegative constants which will be exactly defined in the particular cases
$*$ = the complex conjugation.

In the case when $g(x) \equiv h(x)$ the functional (7) naturally induces a seminorm $\|g\|$ of the function $g(x)$

$$\|g\| = \sqrt{I(g, g)} \tag{8}$$

in the subspace $\tilde{W}$ which is defined by

$$\tilde{W} = \{g \in W;\ \|g\| < \infty\}. \tag{9}$$

The seminorm (8) will be called the smooth seminorm (SS) because in certain sense it represents the measure of smoothness of the given function $g(x)$. Furthermore we suppose that there exists a complete orthogonal set of functions in $\tilde{W}$ where the orthogonality is understood in a sense of the inner product (7). We denote this set $\{g_\alpha(x)\}$. Now we will use the minimization of the SS as an additional constrain to the interpolation function. The variational formulation of the interpolation problem is then as follows:

Find an interpolation function $S(x)$ which fulfils the conditions

(a) $$S(x^{(j)}) = z^{(j)}, \quad j = 1, \ldots, N, \tag{10}$$

(b) $$\|S\| = \text{minimum}. \tag{11}$$

The unique solution of conditions (10) and (11) according to Talmi and Gilat [12] is given by

$$S(x) = T(x) + \sum_{j=1}^{N} \lambda_j R(x, x^{(j)}), \tag{12}$$

where

$$T(x) = \sum_{l=1}^{M} a_l f_l(x) \tag{13}$$

and $\{f_l(x)\}$ is a set of linearly independent functions (monomials) which have zero SS, so that

$$\|f_l\| = 0, \quad l = 1, \ldots, M, \tag{14}$$

while the generating function (GF) $R(x, y)$ is given by

$$R(x, y) = \sum_{\alpha} \frac{g_\alpha^*(x) g_\alpha(y)}{\|g_\alpha\|^2}. \tag{15}$$

The function $T(x)$ will be called the trend function, although it is not the trend in the usual statistical sense. The coefficients $\{a_l\}$ in equation (13) and $\{\lambda_j\}$ in equation (12) are found by the solution of the system of linear equations

$$\sum_{j=1}^{N} \lambda_j f_l(x^{(j)}) = 0, \qquad l = 1, \ldots, M, \tag{16}$$

$$S(x^{(j)}) = z^{(j)}, \quad j = 1, \ldots, N. \tag{17}$$

It is useful to note that an arbitrary function $G(x, y)$ can be added to the GF if there exist such constants $\{b_l\}$ that the following equation holds:

$$\sum_{j=1}^{N} \lambda_j G(x, x^{(j)}) = \sum_{l=1}^{M} b_l f_l(x). \tag{18}$$

Of course, by simple redefining of equation (13)

$$T(x) = \sum_{l=1}^{M} (a_l - b_l) f_l(x) \tag{19}$$

the interpolating function $S(x)$ is not changed. We will use this property later.

It is clear that the solution (12)–(17) is rather formal. Our aim is to show that under some general conditions it is possible to find the representation of the GF which essentially leads to explicit results for $T(x)$ and $R(x, y)$.

We will continue with a proposal that the interpolating function we are looking for is periodic in all variables with period $L (\Omega = L^d)$ so that the following equation holds:

$$S(x) = S(x + vL), \tag{20}$$

where $v = (v_1, v_2, \ldots, v_d)$ is an arbitrary vector with integer components. The natural choice for the set of orthogonal functions is

$$g_k(x) = L^{-d/2} \, \mathrm{e}^{-ix \cdot k}, \quad k = \frac{2\pi}{L} v, \tag{21}$$

($x \cdot k$ means, as usually, the scalar product of the vectors x and k, i denotes the imaginary unit). In this case

$$R(x, y) = \frac{1}{L^d} \sum_k \frac{\mathrm{e}^{-ik \cdot (x-y)}}{\| g_k \|^2}. \tag{22}$$

Now we enlarge the region Ω to the whole space by assuming the limit $L \to \infty$. In this limit the components of k change from discrete to continuous variables which induces the following correspondences:

$$L \to \infty$$

$$\frac{1}{L^d} \sum_k \to \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathrm{d}k, \quad \mathrm{d}k = \mathrm{d}k_1, \mathrm{d}k_2 \ldots \mathrm{d}k_d, \quad k \in \mathbb{R}^d \tag{23}$$

$$\| g_k \|^2 \to \sigma(k) = \sum_\alpha B_\alpha k_1^{2\alpha_1} k_2^{2\alpha_2} \ldots k_d^{2\alpha_d} \tag{24}$$

and the Fourier sum (22) changes to the following Fourier integral:

$$R(x, y) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{\mathrm{e}^{-ik \cdot (x-y)}}{\sigma(k)} \mathrm{d}k \tag{25}$$

and this is the desired tractable form of the GF. Moreover, in all cases which will be considered below, we assume that the space in which the interpolation is performed is isotropic. This condition makes useful the following notations:

$$q = (k_1^2 + k_2^2 + \cdots + k_d^2)^{1/2} \tag{26}$$

$$r = [(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_d - y_d)^2]^{1/2} \tag{27}$$

$$C(\alpha) = \frac{|\alpha|!}{\alpha_1!\alpha_2!\ldots\alpha_d!}, \tag{28}$$

which will be used later.

There is still a large degree of freedom for the choice of the coefficients $\{B_\alpha\}$ in equation (24). This allows us to find $R(x, y)$ and $T(x)$ in a closed form for some important cases as will be shown in the following sections.

4. SPLINES

In this section we will show the correspondence of the general results from the previous section to the well known spline interpolation methods. Let us choose the SS in the form which includes only the second derivatives. The constants $\{B_\alpha\}$ are given by

$$B_\alpha = \begin{cases} C(\alpha), & |\alpha| = 2, \\ 0, & \text{otherwise.} \end{cases} \tag{29}$$

The trends for the particular dimensions are given by

$$T(x) = a_1 + a_2 x_1, \qquad d = 1\ (M = 2), \tag{30}$$

$$T(x) = a_1 + a_2 x_1 + a_3 x_2, \qquad d = 2\ (M = 3), \tag{31}$$

$$T(x) = a_1 + a_2 x_1 + a_3 x_2 + a_4 x_3, \quad d = 3\ (M = 4), \tag{32}$$

while

$$\sigma(k) = q^4 = (k_1^2 + k_2^2 + \cdots + k_d^2)^2. \tag{33}$$

A special care should be taken to the evaluation of the GF. A simple substitution of equation (33) to equation (25) leads to an unbounded integral. However, by using equation (16) we can see that

$$\sum_{j=1}^{N} \lambda_j = 0 \tag{34}$$

and moreover that

$$\sum_{j=1}^{N} \lambda_j (x - x^{(j)}) \cdot (x - x^{(j)}) = \sum_{j=1}^{N} \lambda_j \sum_{m=1}^{d} (x_m^{(j)})^2 = \text{constant}, \tag{35}$$

is independent on x and the resulting constant in equation (35) can be simply absorbed to a_1. Now if we consider the properties of $R(x, y)$ together with equations (18) and (19) it is possible to rewrite the GF in the form which is finite for all x, y

$$R(x, y) = \frac{1}{2\pi} \int_{\mathbf{R}} \frac{\mathrm{e}^{-ik\cdot(x-y)} - 1 + \frac{1}{2}[k \cdot (x - y)]^2}{q^4} \mathrm{d}k, \qquad d = 1, \tag{36}$$

$$R(x, y) = \frac{1}{4\pi^2} \int_{q<u} \frac{\mathrm{e}^{-ik\cdot(x-y)} - 1 + \frac{1}{2}[k \cdot (x - y)]^2}{q^4} \mathrm{d}k + \frac{1}{4\pi^2} \int_{u \leqslant q} \frac{\mathrm{e}^{-ik\cdot(x-y)} - 1}{q^4} \mathrm{d}k, \quad 0 < u < \infty, \quad d = 2, \tag{37}$$

$$R(x, y) = \frac{1}{8\pi^3} \int_{\mathbf{R}^3} \frac{\mathrm{e}^{-ik\cdot(x-y)} - 1}{q^4} \mathrm{d}k, \qquad d = 3. \tag{38}$$

(In fact, this step is not completely correct in the presented manner but it can be rigorously justified within the framework of generalized functions. However, this aspect is out of the scope of this paper.) For $d = 1$ and $d = 3$ the integrals (36) and (38) are simple

$$R(x, y) = R(r) = \frac{r^3}{12}, \quad d = 1, \tag{39}$$

$$R(x, y) = R(r) = -\frac{r}{8\pi}, \quad d = 3. \tag{40}$$

For the two-dimensional case it is convenient to rewrite equation (37) to the form

$$R(x, y) = \frac{1}{4\pi^2} \lim_{\epsilon \to 0} \left[\int_{a < u} \frac{\mathrm{e}^{-ik \cdot (x-y)} - 1 + \frac{1}{2}[k \cdot (x - y)]^2}{q^4 + \epsilon^4} \mathrm{d}k + \int_{u \leqslant q} \frac{\mathrm{e}^{-ik \cdot (x-y)} - 1}{q^4 + \epsilon^4} \mathrm{d}k \right]. \tag{41}$$

After the integration [15] one obtains

$$R(x, y) = R(r) = \lim_{\epsilon \to 0} \frac{1}{2\pi} \left\{ -\frac{1}{\epsilon^2} \mathrm{kei}(\epsilon r) + \frac{\pi}{4\epsilon^2} + \frac{r^2}{16} \left[\ln(u^{1/4} + \epsilon^4) - \ln \epsilon^4 \right] \right\}, \tag{42}$$

where kei$(\cdot)$ is the Kelvin function [16]. Using the series expansion of this function we can finally perform the limit with the result

$$R(r) = \frac{1}{8\pi} r^2 \ln r + cr^2 \tag{43}$$

where c is a certain constant. The second term here is not important because it can be absorbed to $T(x)$ due to condition (35) and thus

$$R(r) = \frac{1}{8\pi} r^2 \ln r, \; d = 2. \tag{44}$$

As one expects the one-dimensional result is simply the spline of the third order [6]. Similarly, for $d = 2$ we have obtained the interpolation function which is known as the thin plate spline developed by Duchon [7] and successfully applied, for example, by Dubrule [17] and Franke [18]. While these two results (39) and (44) are often used in practice the three-dimensional case is not very useful because the first (partial) derivatives are divergent at the data points (for $d = 2$ this unpleasant effect appears in second derivatives, for $d = 1$ in the third one). In Sections 6 and 7 we will show how to regularize these divergencies by the inclusion of the higher derivatives in the SS.

5. SPLINES WITH TENSION

According to our discussion in Section 2 there are some cases when the spline method can be considerably improved by the inclusion of the first derivatives (a tension or membrane term) into the SS. Thus,

$$B_\alpha = \begin{cases} C(\alpha), & |\alpha| = 2, \\ \varphi^2 C(\alpha), & |\alpha| = 1, \\ 0, & \text{otherwise,} \end{cases} \tag{45}$$

where $\varphi > 0$ represents the weight of the tension term in the SS. The trend function is given by

$$T(x) = a_1 \tag{46}$$

for all dimensions and further

$$\sigma(k) = \varphi^2 q^2 + q^4. \tag{47}$$

In order to keep the GF finite it should be modified by the arguments analogical to that in the previous section with the result

$$R(r) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{\mathrm{e}^{-ik \cdot (x-y)} - 1}{\varphi^2 q^2 + q^4} \mathrm{d}k, \quad d = 1, 2. \tag{48}$$

The expressions for the GF's are

$$R(r) = \frac{1}{2\varphi^3}(1 - \varphi r - e^{-\varphi r}),\ d = 1, \tag{49}$$

$$R(r) = -\frac{1}{2\pi\varphi^2}\left[\ln\left(\frac{r\varphi}{2}\right) + c_E + K_0(r\varphi)\right],\quad d = 2, \tag{50}$$

where a somewhat more complicated case for $d = 2$ has been derived by rewriting equation (48) to the form

$$R(r) = \lim_{\epsilon \to 0} \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \frac{e^{-ik\cdot(x-y)} - 1}{(q^2 + \epsilon^2)(q^2 + \varphi^2)}\, dk,\quad d = 2. \tag{51}$$

In equation (50), $K_0(\cdot)$ is the modified Bessel function of the zeroth order and $c_E = 0.577215\ldots$ is the Euler constant. For the practical purposes there exist relatively simple and accurate polynomial-like approximations of the Bessel functions in Abramowitz and Stegun [16]. We have not presented the result for $d = 3$ because of the deficiency of the GF which is the same as that one in the Section 4.

The results of this section offer the possibility to tune the character of the interpolant from a simple spline ($\varphi \to 0$) to the membrane-like shape ($\varphi \to \infty$) according to the interpolated phenomenon.

6. REGULARIZED SPLINES

In order to remove the singularities of the derivatives at the data points (what is severe especially for $d = 2, 3$) we propose to include the third derivatives into the SS

$$B_\alpha = \begin{cases} C(\alpha), & |\alpha| = 2, \\ \tau^2 C(\alpha), & |\alpha| = 3, \\ 0, & \text{otherwise}. \end{cases} \tag{52}$$

The parameter $\tau > 0$ obviously measures the weight of the term with the third derivatives in the SS. The trends are given by equations (30)–(32) and by arguments essentially same to that in Section 4 we can write the GF in the form

$$R(r) = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \frac{e^{-ik\cdot(x-y)} - 1 + \frac{1}{2}[k\cdot(x-y)]^2}{q^4 + \tau^2 q^6}\, dk,\quad d = 2, \tag{53}$$

$$R(r) = \frac{1}{8\pi^3} \int_{\mathbb{R}^3} \frac{e^{-ik\cdot(x-y)} - 1}{q^4 + \tau^2 q^6}\, dk,\quad d = 3. \tag{54}$$

The resulting expressions are

$$R(r) = \frac{1}{2\pi}\left\{\frac{r^2}{4}\left[\ln\left(\frac{r}{2\tau}\right) + c_E - 1\right] + \tau^2\left[K_0\left(\frac{r}{\tau}\right) + c_E + \ln\left(\frac{r}{2\pi}\right)\right]\right\},\quad d = 2, \tag{55}$$

$$R(r) = \frac{\tau^2}{4\pi r}\left[e^{-r/\tau} - 1 + \frac{r}{\tau} - \frac{r^2}{2\tau^2}\right],\quad d = 3. \tag{56}$$

For $d = 2$ it can be checked by the series expansion of $K_0(\cdot)$ that the GF has regular partial derivatives everywhere up to the third order. For $d = 3$ this is the case for the derivatives up to the second order. We recall that the explicit expressions for the derivatives of $K_0(\cdot)$ can be found in terms of $K_0(\cdot)$ and $K_1(\cdot)$ by using the corresponding recurrent relations from the theory of Bessel functions (see, for example, Abramowitz and Stegun [16]). Finally, by the inspection of equation (55) we can see that for $\tau \ll 1$ the change of the GF when comparing to the spline case [equation (44)] is significant only for $r \ll 1$. This implies that, as we have mentioned in the Section 2, the inclusion of third derivatives with the small weight changes the interpolation function only in the small neighbourhoods of the data points.

7. REGULARIZED SPLINES WITH TENSION

In this section we will derive a rather general interpolation functions which, in a certain way, synthetize the advantages of the previous cases—the control of the interpolant's tension with the preserved regularity of the derivatives of sufficiently high order. For $d = 1$ and $d = 3$ it is even possible to obtain the explicit results with all derivatives in the SS in a manner proposed by Talmi and Gilat [12]

$$B_\alpha = \begin{cases} 0, & |\alpha| = 0, \\ C(\alpha)D^{2|\alpha|}, & \text{otherwise.} \end{cases} \tag{57}$$

The relative weight of the particular members in the SS series is controlled by the parameter D. The trend functions are given by equation (46) and the GF by

$$R(r) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{e^{-ik\cdot(x-y)} - 1}{\cosh(kD) - 1} \, dk, \quad d = 1, 3. \tag{58}$$

It is possible to obtain the GF's in the closed form

$$R(r) = -\frac{r}{D^2} \coth\left(\frac{\pi r}{D}\right), \quad d = 1, \tag{59}$$

$$R(r) = \frac{1}{2\pi^2 D^2} \frac{1}{r} \frac{\partial}{\partial r} \left[r \coth\left(\frac{\pi r}{D}\right)\right], \quad d = 3. \tag{60}$$

In two dimensions the situation is more complicated and we did not succeed in obtaining the explicit result with the choice (57). However, we can modify the SS as follows:

$$B_\alpha = \begin{cases} D^2\varphi^2 C(\alpha), & |\alpha| = 1, \\ D^4 C(\alpha), & |\alpha| = 2, \\ D^6 \tau^2 C(\alpha), & |\alpha| = 3, \\ 0, & \text{otherwise.} \end{cases} \tag{61}$$

While the trend function is the same as for $d = 1, 3$ by inspection of equations (34) and (48) one finds that the GF is given by

$$R(r) = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \frac{e^{-ik\cdot(x-y)} - 1}{\varphi^2(Dq)^2 + (Dq)^4 + \tau^2(Dq)^6} \, dk. \tag{62}$$

The integration can be carried out by the decomposition of the fraction into the partial ones and then by manipulations similar to those used in Section 4 we have

$$R(r) = \frac{1}{D\varphi^2} \left\{ -\ln\left(\frac{r}{2D}\right) - c_E + \frac{1}{v - w} \left[wK_0\left(\frac{r}{D}\sqrt{v}\right) - vK_0\left(\frac{r}{D}\sqrt{w}\right) + \frac{w}{2}\ln v - \frac{v}{2}\ln w \right] \right\}, \tag{63}$$

where

$$v = \frac{1}{2\tau^2}\left(1 + \sqrt{1 - 4\varphi^2\tau^2}\right), \quad w = \frac{1}{2\tau^2}\left(1 - \sqrt{1 - 4\varphi^2\tau^2}\right). \tag{64}$$

From conditions (64) there is an obvious condition for φ, τ

$$1 - 4\varphi^2\tau^2 > 0, \tag{65}$$

but otherwise their choice is arbitrary. However, in order to avoid too many parameters, we can set for instance $\varphi^2 = \tau^2 = 10^{-2}$ and thus the properties of the interpolation function is dependent solely on D. Clearly, if $D \approx 1$ the term with second derivatives has the largest relative weight in the SS. On the other hand, if $D \ll 1$ ($D \gg 1$) the term with the first (third) derivatives becomes to be the dominant one. Finally we note that for $d = 2$ the derivatives are regular everywhere up to the third order while for $d = 1, 3$ this is the case for the derivatives of all orders.

8. DISCUSSION AND CONCLUSIONS

In this paper we have presented some results of the general variational approach to the interpolation problem. Within this treatment we have shown that some known spline interpolants are its special cases. We have also derived new interpolants with several useful properties. The regularized spline is designed to yield simultaneously good approximations both for the function itself and its derivatives as well. The spline with tension admits to tune the smoothness of the interpolant according to the character of modelled phenomenon. The natural synthesis of both cases is the regularized spline with tension. The flexibility of the interpolation functions is interpreted within a simple mechanical model which we consider rather instructive for the applications.

Among the most useful properties of the presented methods is that no complicated numerical procedures are required. The interpolant is given in its explicit form and the only task is to compute the coefficients in equation (12) from the system of linear equations (16) and (17). Consequently also the derivatives are known explicitly what is rather important for the commonly required estimation of gradients, curvatures, topographic structures especially in two-dimensional cases.

Concerning the computer processing it should be mentioned that the methods may seem to be uneffective for large data sets as the system of $\sim N$ equations should be solved (the methods are global in the sense of Franke's [3] definition). However, this drawback can be overcome by segmented processing proposed by Franke [18] which is not very complicated and makes the method effective indeed. Another important feature is that the machinery is the same for the arbitrary dimension because going from one dimension to another the only change is the substitution of the appropriate expressions for $R(\cdot)$ and $T(\cdot)$ to equation (12).

However, by this the whole potential of the approach is not exhausted and the methods can be further developed. For example, Talmi and Gilat [12] have proposed the following extensions. They showed the natural way to incorporate the prescribed values of derivatives, integrals or other linear conditions into the interpolation function. Moreover, they proposed a straightforward modification of the method to the problem of smoothing so it could be applied also to noisy data.

We illustrate the result of this approach by the following test example. The regularized spline was used for the interpolation of the bivariate function proposed by Franke [3] and used in tests by Renka and Cline [19] and Foley [20]. The function is sampled by scattered 100 data points within the unit square (we have digitized their actual positions from the Fig. 5. in Renka and Cline [19]). The accuracy of the interpolation is tested at 1089 grid points of a 33×33 uniform mesh. The comparison of the results with some other methods including those based on the C^1 interpolants on the triangular networks is given in Table 1 (the first part of it is taken from Renka and Cline [19] and represents some of the methods which scored among the best in Franke's tests [3]). In this comparison the regularized spline gives significantly better results. Moreover, this is true over the relatively wide range of τ values, so at least in this case the interpolant is rather stable with respect to changes of this parameter.

However, the question of the choice of the parameters is a rather important point for discussion. We consider the free parameter(s) in the interpolation function to be a valuable tool for the control of the interpolant's character. We believe that it will be possible to find a procedure for the

Table 1. Comparison of the mean and maximum absolute errors for various methods of bivariate interpolation

Method	Mean error	Maximum error
Akima Mod. III	0.00729	0.0520
Mod. quadr. Shepard	0.00785	0.0573
Lawson	0.00783	0.0951
Renka global	0.00540	0.0499
Renka local	0.00619	0.0505
Nielson–Franke quadr.	0.00741	0.0782
Nielson min. norm	0.00537	0.0492
Thin plate spline	0.00497	0.0470
Regularized spline $\tau^2 = 0.5$	0.00222	0.0259
Regularized spline $\tau^2 = 0.1$	0.00207	0.0234
Regularized spline $\tau^2 = 0.01$	0.00227	0.0233
Regularized spline $\tau^2 = 0.001$	0.00324	0.0274

automatic choice of optimal values (for example in the statistical sense) for the parameter(s) and this problem is currently being studied. Our aim is to minimize the empirical inputs which on one hand can give excellent results [20] but on the other hand they require considerable "trial and error" processing.

In conclusion, we want to point out that this paper can be understood also as a methodological outline how to construct the interpolant with the desired properties within the possibilities of the variational approach.

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